limit theorems for SDEs with irregular drifts

Jianhai Bao

Tianjin University

Outline

- Motivations
- LLN and CLT: SDEs with Hölder drifts
- LLN and CLT: SDEs with discontinuous drifts
- Main results

This talk is based on a joint work with **Jiaqing Hao**.

Motivations

- $(X_t)_{t\geq 0}$: a Markov process; μ: the IPM;
- Additive functional: $\int_0^t f(X_s)\,\mathrm{d}s$, f: observable.
  - SLLN: $\frac{1}{t}\int_0^t f(X_s)\,\mathrm{d}s \to \mu(f)$ a.s.
  - CLT: $\frac{1}{\sqrt{t}}\int_0^t (f(X_s)-\mu(f))\,\mathrm{d}s$ converges weakly a normal r.v.
- Methods:
  - moment method (Markov (1908));
  - Bernstein's block method (Bernstein (1926));
  - martingale method+Poisson equation (Papanicolaou, et al. (1977));
- Applications: e.g. large deviations and moderate deviations;

Motivations

Limit theorems:

- Nondegenerate setting: Ergodicity in the total variation + mixing condition, e.g., Kulik'18; L^2-ergodicity;
- Degenerate setup: ergodicity under the (quasi)-Wasserstein distance, e.g.
  - functional SDE (Itô-Nisio'64): $\mathrm{d}X(t) = b(X_t)\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}W(t)$;
  - 2D N-S equations with degenerate stochastic forcing.

Motivations

Existing works for SDEs/SPDEs with regular coefficients:

- Weak LLN + CLT (Komorowskia and Walczuk, 2012):
  - Observable: Lipschitz;
  - $\mathbb{W}_1$-exponential contractivity;
  - Dissipative SDEs with regular drifts;
  - Convergence rate is unavailable.
- General framework (Shirikyan'06):
  - Strong LLN: uniform mixing & uniform moment estimate;
  - CLT: uniform mixing & uniform exponential estimates;

Motivations

- Exponential ergodicity of functional SDE via Wasserstein coupling (B.-Wang-Yuan, 2020a);
- Convergence rate of LLN and CLT (B.-Wang-Yuan, 2020b);
- Convergence rate of LLN and CLT (Wang-Wu-Zhu, 2020);
- Numerical LLN and CLT:
  - finite dimension: forward EM (Pagès-Panloup'12, Lu-Tan-Xu'21), backward EM (Jin'23);
  - infinite dimension: full discretization of parabolic SPDEs (Chen' et al.'23)

Our goals:

- Improve convergence rate;
- Drop exponential estimate;
- Cover SDEs with irregular drifts.

- SDE:

$$\mathrm{d}X_t = \big(b_0(X_t) + b_1(X_t)\big)\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}W_t. \tag{1}$$

Assume that

$(\mathbf{H}_b)$ $b_1 : \mathbb{R}^d \to \mathbb{R}^d$ is locally Lipschitz and $\exists\ \lambda_1, \lambda_2, \ell_0 > 0$ such that

$$2\langle x-y, b_1(x)-b_1(y)\rangle \leq \lambda_1|x-y|^2\mathbf{1}_{\{|x-y|\leq\ell_0\}}-\lambda_2|x-y|^2\mathbf{1}_{\{|x-y|\geq\ell_0\}},$$

$b_0 \in C^\alpha(\mathbb{R}^d)$ for some $\alpha \in (0,1)$, i.e., $\exists\ K_1 > 0$ such that

$$|b_0(x) - b_0(y)| \leq K_1|x-y|^\alpha, \qquad x, y \in \mathbb{R}^d.$$

LLN: SDEs with Hölder continuous drifts

Assume that

$(\mathbf{H}_\sigma)$ $\sigma : \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^d$ is Lipschitz continuous, i.e., $\exists\ K_2 > 0$ such that

$$\|\sigma(x) - \sigma(y)\|_{\mathrm{HS}}^2 \leq K_2 |x - y|^2$$

and $\exists\ \kappa \geq 1$ such that

$$\frac{1}{\kappa}|y|^2 \leq \langle (\sigma\sigma^*)(x)y, y\rangle \leq \kappa |y|^2.$$

Under $(\mathbf{H}_b)$ and $(\mathbf{H}_\sigma)$, (1) has a unique strong solution.

- Local solution via Zvonkin transformation (Xie, et al.'20, Zhang-Yuan'21);
- Global solution via life time = T.

LLN: SDEs with Hölder continuous drifts

Theorem

Assume $(\mathbf{H}_b)$ *and* $(\mathbf{H}_\sigma)$ *with* $\lambda_2 > (K_2\kappa^3)^{\frac{1}{2}}$*. Then, for* $f \in C_{\mathrm{Lip}}(\mathbb{R}^d)$ *and* $\varepsilon \in (0, 1/2)$*, there exist a random time* $T_\varepsilon \geq 1$ *and a constant* $C > 0$ *s.t. for all* $t \geq T_\varepsilon$,

$$\left| \frac{1}{t} \int_0^t f(X_s^x)\,\mathrm{d}s - \mu(f) \right| \leq Ct^{-\frac{1}{2}+\varepsilon}.$$

LLN: SDEs with Hölder continuous drifts

- When the weight function is constant (Shirikyan'06), the observable is bounded. In our setting, the observable is unbounded.
- In Theorem 2.3 (Shirikyan'06), the convergent rate is $t^{-\frac{1}{2}+r_v}$ for $r_v := q \vee \frac{1+v}{4p}$ with any $q < \frac{1}{2}$ and $v \in (0, 2p-1)$
- In our scenario, the convergence rate is $t^{-\frac{1}{2}+r_v}$, in which $r_v := \frac{1+v}{2p}$ for $v \in (0, p/2-1)$.

$\mathbb{W}_1$-exponential contractivity

Proposition

Assume $(\mathbf{H}_b)$ *and* $(\mathbf{H}_\sigma)$ *with* $\lambda_2 > (K_2\kappa^3)^{\frac{1}{2}}$*. Then, there exist constants* $C^*, \lambda^* > 0$ *such that for all* $t \geq 0$ *and* $\mu, \nu \in \mathcal{P}_1(\mathbb{R}^d)$*,*

$$\mathbb{W}_1(\mu P_t, \nu P_t) \leq C^* \mathrm{e}^{-\lambda^* t} \mathbb{W}_1(\mu, \nu).$$

- In [Theorem 3.1, Wang'23]: abstract framework based on the reflection coupling traced back to Priola-Wang' 06;
- In [Theorem 1.3, Luo-Wang'16], $\mathbb{W}_p$-exponential decay.

$\mathbb{W}_1$-exponential contractivity

- Auxiliary SDE: for $\widetilde{\sigma}(x)^2 := (\sigma\sigma^*)(x) - \frac{1}{\kappa} I_{d\times d}$,

$$\mathrm{d}Y_t = b(Y_t)\mathrm{d}t + \widetilde{\sigma}(Y_t)\mathrm{d}\widetilde{W}_t + \frac{1}{\sqrt{\kappa}}\mathrm{d}\widehat{W}_t.$$

- Coupling SDE

$$\begin{cases} \mathrm{d}\widehat{Y}_t = b(\widehat{Y}_t)\mathrm{d}t + \widetilde{\sigma}(\widehat{Y}_t)\mathrm{d}\widetilde{W}_t + \frac{1}{\sqrt{\kappa}}\Pi_{Z_t}\mathrm{d}\widehat{W}_t, & t < \tau, \\ \mathrm{d}\widehat{Y}_t = b(\widehat{Y}_t)\mathrm{d}t + \widetilde{\sigma}(\widehat{Y}_t)\mathrm{d}\widetilde{W}_t + \frac{1}{\sqrt{\kappa}}\mathrm{d}\widehat{W}_t, & t \geq \tau. \end{cases}$$

- Test function (piecewise C^2-function):

$$f(r) = \frac{\kappa}{2}\int_0^r \mathrm{e}^{-\frac{\kappa}{4}\int_0^u \phi(v)\,\mathrm{d}v}\int_u^\infty s\mathrm{e}^{\frac{\kappa}{4}\int_0^s \phi(v)\,\mathrm{d}v}\,\mathrm{d}s, \quad r \geq 0,$$

where for $\hbar_0 := \ell_0 \vee \big((4K_1)/(\lambda_2 - (K_2\kappa^3)^{\frac{1}{2}})\big)^{\frac{1}{1-\alpha}}$,

$$\phi(u) := \big((\lambda_1 + \lambda_2)u + 2K_1u^\alpha\big)\mathbf{1}_{\{u\leq \hbar_0\}} - \frac{1}{2}\big(\lambda_2 - (K_2\kappa^3)^{\frac{1}{2}}\big)u.$$

CLT: SDEs with Hölder continuous drifts

Assume $(\mathbf{H}_\sigma)$ and

$(\mathbf{H}_b')$ $b_0 \in C^\alpha(\mathbb{R}^d)$ satisfying (7) and $\exists\ \lambda, \lambda^* > 0$ such that

$$\langle x-y, b_1(x)-b_1(y)\rangle \le \lambda |x-y|^2$$

and

$$\langle x, b_1(x)\rangle \le -\lambda^*|x|^2 + C_{\lambda^*}.$$

CLT: SDEs with Hölder continuous drifts

- Functional class: for $p \geq 2$ and $\theta \in (0,1]$, $C_{p,\theta}(\mathbb{R}^d)$ is the collection of all continuous functions $f : \mathbb{R}^d \to \mathbb{R}$ such that

$$\|f\|_{p,\theta} := \sup_{x\in\mathbb{R}^d} \frac{|f(x)|}{1+|x|^p} + \sup_{x\neq y, x,y\in\mathbb{R}^d} \frac{|f(x)-f(y)|}{\psi_{p,\theta}(x,y)},$$

where

$$\psi_{p,\theta}(x,y) := (1\wedge|x-y|^\theta)(1+|x|^p+|y|^p).$$

- Corrector:

$$R_f(x) = \int_0^\infty \big((P_t f)(x) - \mu(f)\big)\,\mathrm{d}t, \quad x\in\mathbb{R}^d.$$

- Characterization of asymptotic variance:

$$\varphi_f(x) := \mathbb{E}\Big|\int_0^1 f(X_r^x)\,\mathrm{d}r + R_f(X_1^x) - R_f(x)\Big|, \qquad x\in\mathbb{R}^d.$$

CLT: SDEs with Hölder continuous drifts

Theorem

Assume $(\mathbf{H}_b')$ *and* $(\mathbf{H}_\sigma)$. *Then, for any* $f \in C_{p,\theta}(\mathbb{R}^d)$ *with* $\mu(f) = 0$, $\sigma := \mu(\varphi_f) \geq 0$ *and* $\varepsilon \in (0, \frac{1}{4})$, $\exists\, C_0 = C_0(\|f\|_{p,\theta}, \sigma, |x|) > 0$ *such that*

$$\sup_{z \in \mathbb{R}^d} \Big(\theta_\sigma(z) \Big|\mathbb{P}(A_t^{f,x} \leq z) - \Phi_\sigma(z)\Big|\Big) \leq C_0 t^{-\frac{\varepsilon}{4}+\varepsilon}, \quad t \geq 1,$$

where $\theta_\sigma(z) := \mathbf{1}_{\{0<\sigma<\infty\}} + (1 \wedge |z|)\mathbf{1}_{\{\sigma=0\}}$.

$\mathbb{W}_{\psi_{p,\theta}}$-exponential contractivity

Proposition

Under $(\mathbf{H}_b')$ *and* $(\mathbf{H}_\sigma)$*, for any* $p \geq 2$*,* $\theta \in (0,1]$ *and* $\mu, \nu \in \mathcal{P}_{\psi_{p,\theta}}(\mathbb{R}^d)$*,* $\exists$ $C^* \geq 1, \lambda^* > 0$ *such that*

$$\mathbb{W}_{\psi_{p,\theta}}(\mu P_t, \nu P_t) \leq C^* \mathrm{e}^{-\lambda^* t} \mathbb{W}_{\psi_{p,\theta}}(\mu, \nu), \qquad t \geq 0,$$

where

$$\psi_{p,\theta}(x,y) := \big(1 \wedge |x-y|^\theta\big)\big(1 + |x|^p + |y|^p\big).$$

$\mathbb{W}_{\psi_{p,\theta}}$-exponential contractivity

- Key inequality:

$$\frac{1}{2}f'(r)(\lambda r + K_1 r^{\alpha}) + \frac{2}{\kappa}f''(r) \le -\frac{1}{2}r^{\theta}, \qquad r \in (0, l_p^*].$$

- Test function

$$f(r) = c^*(r \wedge l_p^*)^{\theta} + h(r \wedge l_p^*), \quad r \ge 0.$$

- A direct calculation shows that

$$\frac{1}{2}h'(r)(\lambda r + K_1 r^{\alpha}) + \frac{2}{\kappa}h''(r) = -r^{\theta}, \qquad r \in (0, l_p^*]$$

and

$$\begin{aligned}&\frac{1}{2}f'(r)(\lambda r + K_1 r^{\alpha}) + \frac{2}{\kappa}f''(r)\\ &= c^*\theta\Big(\frac{1}{2}\big(\lambda r^{\theta} + K_1 r^{\theta+\alpha-1}\big) - \frac{2}{\kappa}(1-\theta)r^{\theta-2}\Big) - r^{\theta}.\end{aligned}$$

CLT: SDEs with Hölder continuous drifts

Proposition

(Proposition 2.10, Shirikyan'0) For a zero-mean martingale (M_k), $\exists\ \beta, B > 0$ *such that*

$$\mathbb{E}e^{|M_k - M_{k-1}|^\beta} \le B, \qquad 1 \le k \le n.$$

Then, for any $\widetilde{\sigma} > 0$ *and* $\varepsilon \in (0, 1/4)$, $\exists\ A_\varepsilon(\widetilde{\sigma}) > 0$ *such that for any* $q > 0$,

$$\sup_{z\in\mathbb{R}} \left|\triangle_\sigma(n^{-\frac{1}{2}} M_n, z)\right| \le A_\varepsilon(\widetilde{\sigma}) n^{-\frac{1}{4}+\varepsilon} + \sigma^{-4q} n^{q(1-4\varepsilon)} \mathbb{E}|n^{-1}V_n^2 - \sigma^2|^{2q}.$$

SDEs with discontinuous drifts

- Scalar SDE:

$$\mathrm{d}X_t = b(X_t)\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}W_t.$$

- Assume that

(A_1) $\exists c_1, c_2, p_0 > 0$ s.t.

$$2xb(x) + (p_0 - 1)|\sigma(x)|^2 \leq c_1 - c_2|x|^2.$$

(A_2) $c, p_1, \alpha > 0$, $\xi_0, \cdots, \xi_{k+1} \in [-\infty, +\infty]$ with $-\infty = \xi_0 < \xi_1 < \cdots, < \xi_k < \xi_{k+1} = \infty$ s.t for all $i \in \{1, \cdots, k+1\}$ and all $x, y \in (\xi_{i-1}, \xi_i)$

$$2(x-y)(b(x)-b(y)) + (p_1 - 1)(\sigma(x) - \sigma(y))^2 \leq c(x-y)^2$$

$$|b(x) - b(y)| \leq c\big(1 + |x|^\alpha + |y|^\alpha\big)|x-y|$$

SDEs with discontinuous drifts

Assume that

(A_3) $\beta > 0$ such that

$$|\sigma(x) - \sigma(y)| \leq c\left(1 + |x|^{\beta} + |y|^{\beta}\right)|x - y|.$$

(A_4) $\sigma(\xi) \neq 0$ for all $i \in \{1, \cdots, k\}$.

The transformation

- The transformation (Müller-Gronbach & Yaroslavtseva, 2022):

$$G_{z,\alpha,\nu}(x) = x + \sum_{i=1}^{k} \alpha_i (x - z_i)\phi((x - z_i)\nu),$$

where

- $\alpha := (\alpha_1, \cdots, \alpha_k)$;
- $\phi(x) = (1 - x^2)^4 \mathbf{1}_{[-1,1]}(x)$;
- $\nu \in \rho_{z,\alpha}$ with

$$\rho_{z,\alpha} = \begin{cases} \frac{1}{8|\alpha_1|}, & k = 1 \\ \min\left(\left\{\frac{1}{8|\alpha_i|} : i \in \{1, \cdots, k\} \cup \left\{\frac{1}{2}(z_i - z_{i-1}), i = 2, \cdots, k\right\}\right\}\right), \\ \quad k \geq 2. \end{cases}$$

- $\alpha_i := \frac{b(\xi_i -) - b(\xi_i +)}{2\sigma^2(\xi)}$.

LLN and CLT: SDEs with discontinuous drifts

Theorem

Under (A_1)-(A_4), the LLN (in Theorem 1) and the CLT (in Theorem 3) holds true, respectively.

- Poisson approach: very nice regularity.